Assignment 4
using derived rules

The ten primitive rules in our book form a complete set of rules – that is, for any sequent that is valid, it is possible to construct a proof using just these rules. However, many of these proofs are very long and/or very difficult. Additional rules can be added to our system to make the proofs easier while still preserving the soundness of the rules as long as the rules we add are truth-preserving rules. Our book lists 42 argument forms that are common enough to have regularly used names (pg. 29,30). I find three groups of these to be especially important: Modus Tollens (MT), DeMorgan's Laws (DeM), and Negated Conditional (NegCon – or Neg → in the book). I have already discussed MT in the supplement for Homework 2 and I will discuss the other rules here.

DeMorgan's Laws

DeMorgan's Laws are really a set of laws which interrelate disjunctions and conjunctions. They say that the negation of a disjunction is equivalent to the conjunction of the negation of its disjuncts and the negation of a conjunction is equivalent to the disjunction of the negations of its conjuncts. For proof purposes, the most useful forms of the rules can be presented like this:

~(PvQ) ⊢ ~P&~Q
~P&~Q ⊢ ~(PvQ)
~(P&Q) ⊢ ~Pv~Q
~Pv~Q ⊢ ~(P&Q)

Each of these four sequents is valid and so any substitution instance of them is also valid. For example, each of the following pairs of sentences are equivalent by DeMorgan's Laws:

PvQ	P&Q	~Pv(QvS)	~(P→ ~Q) v ~(P→Q)
~(~P&~Q)	~(~Pv~Q)	~(P&~(QvS))	~((P→ ~Q) & (P→Q))

~(((AvB) → (C&D)) & E)	~(Av(B&D))	~((A→(~BvC))v(A→B))
~((AvB) → (C&D)) v ~E	~A & ~(B&D)	~(A→(~BvC)) & ~(A→B)

EXAMPLE 1 PvQ ⊢ QvP

Step 1. This is an instance of the commutative law of disjunction. To prove it, I notice that the goal is a disjunction. If I was very lucky, I may be able to prove Q or prove P and then use vI, but in general, this strategy will not work. At any rate,

1	(1) PvQ	A
2	(2) ~(PvQ)	A
	? X	

it should be clear in this case that neither Q nor P will follow from our premise. The best strategy for proving a disjunction is typically to use RAA. This means I should assume the opposite of what I am trying to prove and then prove a contradiction.

	? ~X	
	(n) PvQ	RAA

Step 2. Premise two is the negation of a complex sentence. This is a sign that we should use a short-cut rule to simplify our problem. In this case, it is the negation of a disjunction, so we use DeMorgan's Law.

1	(1) PvQ	A
2	(2) ~(PvQ)	A
2	(3) ~P&~Q	2 DeM
2	(4) ~P	3 &E
2	(5) ~Q	3 &E
1,2	(6) Q	1,4 vE
1	(7) PvQ	5,6 RAA(2)

Step 3. Now the proof is easily finished using elimination rules.

1	(1) PvQ	A
2	(2) ~(PvQ)	A
3	(3) ~P&~Q	2 DeM
1,2	(10) P→R	9 →I (3)

EXAMPLE 2 (A&~C)v(~B&D) ├ (Av~B)&(~CvD)

Step 1. The goal is a conjunction, so I will prove each conjunct, and then put them together with &I. Now I have two separate goals. Focusing on the first goal, it is a disjunction, so I will try get it using RAA.

1	(1) (A&~C)v(~B&D)	A
2	(2) ~(Av~B)	A
1	? Av~B	RAA
1	? ~CvD	
1	(n) (Av~B)&(~CvD)	&I

Step 2. Line 2 is now the negation of a disjunction so I will use DeMorgan's to simplify it.

1	(1) (A&~C)v(~B&D)	A
2	(2) ~(Av~B)	A
2	(3) ~A&B	2 DeM
2	(4) ~A	3 &E
2	(5) B	3 &E
1	? Av~B	RAA

1	?	~CvD	
	(n)	(Av~B)&(~CvD)	&I

Step 3. At this point, I am trying to derive a contradiction, but it is not obvious what to aim for. However, I haven't yet used line 1. Line 1 is a disjunction, so to use it, I will use the vE rule. To use vE, I would need to get the negation of one one of the disjuncts. The negation of the first disjuncts is ~(A&~C). This should immediately remind you of DeM. Working backwards, we can see that this is equivalent to ~AvC which we can easily get. It is worth spending some time on this step to make sure you understand it.

1	(1)	(A&~C)v(~B&D)	A
2	(2)	~(Av~B)	A
2	(3)	~A&B	2 DeM
2	(4)	~A	3 &E
2	(5)	B	3 &E
2	(6)	~AvC	4 vI
2	(7)	~(A&~C)	6 DeM
1,2	(8)	~B&D	1,6 vE
1,2	(9)	~B	8 &E
1	(10)	Av~B	5,9 RAA(2)
1	?	~CvD	
	(n)	(Av~B)&(~CvD)	&I

Step 4. Proving the second conjunct of the goal is analogous to proving the first conjunct. So I will simply follow exactly the same procedure.

1	(1)	(A&~C)v(~B&D)	A
2	(2)	~(Av~B)	A
2	(3)	~A&B	2 DeM
2	(4)	~A	3 &E
2	(5)	B	3 &E
2	(6)	~AvC	4 vI
2	(7)	~(A&~C)	6 DeM
1,2	(8)	~B&D	1,6 vE
1,2	(9)	~B	8 &E
1	(10)	Av~B	5,9 RAA(2)
11	(11)	~(~CvD)	A
11	(12)	C&~D	11 DeM
11	(13)	C	12 &E
11	(14)	~D	12 &E
11	(15)	~AvC	13 vI
11	(16)	~(A&~C)	15 DeM
1,11	(17)	~B&D	1,16 vE
1,11	(18)	D	17 &E
1	(19)	~CvD	14,18 RAA(11)
1	(19)	(Av~B)&(~CvD)	10,19 &I

Negated Conditionals

The NegCon (or Neg→) rule says that the negation of a conditional is equivalent to the conjunction of its antecedent and the negation of its consequent. In other words:

$\sim(P \to Q) \vdash P \& \sim Q$ and $P \& \sim Q \vdash \sim(P \to Q)$

EXAMPLE 3 $\sim(P \to (Q \to R)), Q \to S \vdash S \& \sim R$

Step 1. The goal is a conjunction, so I will attempt to prove each part and then put them together with &I. This gives me two goals, S and ~R.

1	(1) ~(P→(Q→R))	A
2	(2) Q→S	A
	? S	
	? ~R	
	(n) S&~R	&I

Step 2. Looking at our goals, neither leads us to an immediate strategy. However, line 1 is the negation of a conditional, so we can simplify it using the NegCon rule.

1	(1) ~(P→(Q→R))	A
2	(2) Q→S	A
1	(3) P&~(Q→R)	1 NegCon
	? S	
	? ~R	
	(n) S&~R	&I

Step 3. Line 2 is now a conjunction which is easy to deal with. The second conjunct is the negation of a conditional, so I will again apply the NegCon rule. This allows me to easily finish the problem.

1	(1) ~(P→(Q→R))	A
2	(2) Q→S	A
1	(3) P&~(Q→R)	1 NegCon
1	(4) P	2 &E
1	(5) ~(Q→R)	2 &E
1	(6) Q&~R	4 NegCon
1	(7) Q	5 &E
1	(8) ~R	5 &E
1,2	(9) S	2,7 →E
1,2	(10) S&~R	8,9 &I

EXAMPLE 4 $(\sim P \to Q) \vee (\sim P \to \sim R) \vdash \sim P \to (Q \vee \sim R)$

Step 1. The goal is a conditional, so I will assume its antecedent and prove its consequent.

1	(1) (~P→Q) v (~P→ ~R)	A
2	(2) ~P	A
	(n-1) Qv~R	new goal
	(n) ~P→(Qv~R)	→I

Step 2. Our new goal is a disjunction, so I will attempt to prove it by RAA. Upon assuming the opposite of my goal, I have a negated disjunction so I will use DeM to simplify it.

1	(1) (~P→Q) v (~P→ ~R)	A
2	(2) ~P	A
3	(3) ~(Qv~R)	A
3	(4) ~Q & R	3 DeM
3	(5) ~Q	4 &E
3	(6) R	4 &E
	? X	
	? ~X	
	(n-1) Qv~R	RAA
	(n) ~P→(Qv~R)	→I

Step 3. I am trying to prove a contradiction but any contradiction will do, so working backwards is difficult. If I look at line 1 which I haven't used yet, I realize that since it is a disjunction, I will have to use vE. So I need the negation of one of the disjuncts. I will arbitrarily try to get the negation of the first one.

1	(1) (~P→Q) v (~P→ ~R)	A
2	(2) ~P	A
3	(3) ~(Qv~R)	A
3	(4) ~Q & R	3 DeM
3	(5) ~Q	4 &E
3	(6) R	4 &E
	? ~(~P→Q)	new goal
	? X	
	? ~X	
	(n-1) Qv~R	RAA
	(n) ~P→(Qv~R)	→I

Step 4. My new goal is the negation of a conditional, so I naturally think of the NegCon rule. According to this rule, I would first need to get ~P&~Q which is easy.
Once I use the NegCon rule, getting the contradiction is not difficult.

1	(1) (~P→Q) v (~P→ ~R)	A
2	(2) ~P	A
3	(3) ~(Qv~R)	A
3	(4) ~Q & R	3 DeM
3	(5) ~Q	4 &E
3	(6) R	4 &E
2,3	(7) ~P&~Q	2,5 &I
2,3	(8) ~(~P→Q)	7 NegCon
1,2,3	(9) ~P→ ~R	1,8 vE
1,2,3	(10) ~R	2,9 →E
1,2	(11) Qv~R	6,10 RAA (3)
1	(12) ~P→(Qv~R)	11 →I(2)